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THE LATERAL MIGRATION OF SOLID PARTICLES IN A LAMINAR FLOW NEAR A PLANE

R. G. Cox and S. K. Hsu

Department of Civil Engineering and Applied Mechanics, McGill University, Montreal, Canada

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Abstract—The lateral inertial migration of a solid spherical particle suspended in a laminar flow over a vertical wall is considered theoretically. Formulae for the migration velocity are obtained for both neutrally buoyant and non-neutrally buoyant particles and also for the case of zero flow over the wall. Situations in which the particle is either free to rotate or prevented from rotating are considered. Such results are found to agree qualitatively with known experimental data.

1. INTRODUCTION

The lateral migration of neutrally buoyant and non-neutrally buoyant solid spherical particles suspended in laminar tube flows has been extensively studied experimentally (Segre & Silberberg 1961, 1962*a, b*; Oliver 1962; Eichhorn & Small 1964; Theodore 1964; Repetti & Leonard 1964; Jeffrey & Pearson 1965; Karnis *et al.* 1966*a, b*; Denson *et al.* 1966; Halow 1968; Yanizeski 1968; Halow & Wills 1970*a, b*; Tachibana 1973), a survey of this work having been presented by Brenner (1966). This migration which results from the effects of fluid inertia was studied theoretically by Rubinov & Keller (1961) and Saffman (1965). However in these studies the effects of the solid bounding walls of the tube and of the variation of the rate of shear across the tube were omitted. However Cox & Brenner (1968) derived expressions for the particle migration in a general tube flow by making a double expansion of the flow field in terms of the Reynolds number and the ratio of particle radius to tube size. These results were not evaluated explicitly but were left in the form of volume integrals involving the Green's function for creeping motion flow in the tube. In the present paper these results are used to calculate analytically the migration velocity of a spherical particle in a flow field near a single vertical plane wall. The situations in which the particle is either free to rotate or prevented from rotating, is either neutrally buoyant or non-neutrally buoyant, as well as the case where there is no flow over the wall are considered. Thus in section 2 a brief description is given of the results obtained by Cox & Brenner (1968) together with a discussion of their applicability to the present problem. In sections 3 and 4 the Green's function for creeping motion flow near the plane wall is found in a form suitable for the calculation of the migration velocity which is done in section 5. Then in the final section the results obtained are discussed and compared with the known experimental results.

2. LATERAL MIGRATION OF A SPHERE

Consider a viscous fluid with viscosity μ and density ρ occupying the semi-infinite region $r_1' > 0$ and bounded by a solid rigid wall W at $r_1' = 0$. At a distance d from the wall, a small sphere of radius a is free to move in the fluid which undergoes a rectilinear flow $\mathbf{U}'(\mathbf{r}')$ in the r_2' direction where

$$\mathbf{U}'(\mathbf{r}') = (0, U_2'(r_1'), 0), \qquad [2.1]$$

the characteristic velocity and length scale of this flow being V and d respectively (see figure 1). Then in terms of the length scale ratio

$$\kappa = a/d, \qquad [2.2]$$

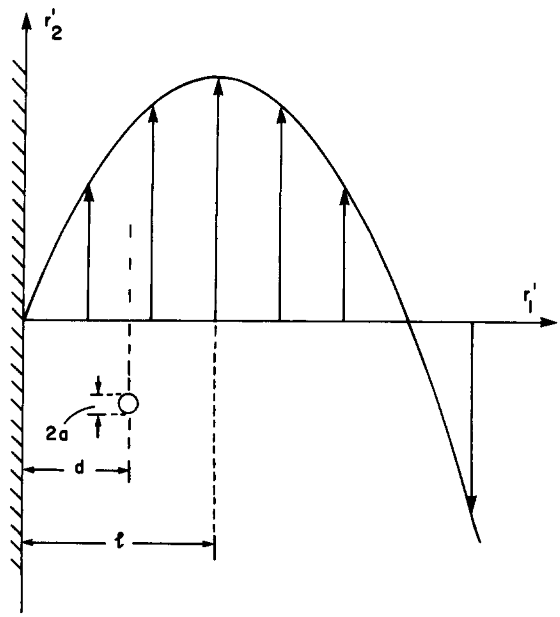

Figure 1. Spherical particle in a fluid flow bounded by a vertical plane wall.

and the Reynolds number

$$Re = aV/\nu \quad \text{where} \quad \nu = \mu/\rho, \qquad [2.3]$$

Cox & Brenner (1968) derived formulae for the migration velocity of the sphere across streamlines (i.e. towards or away from the wall) as the result of fluid inertia. This was done by making a double expansion in the parameters κ and Re both being assumed small. These results (which were derived for a more general system than that discussed here) were not explicitly evaluated but were left in the form of volume integrals. It is the evaluation of these for the present problem which will concern us here.

Since the undisturbed flow field $\mathbf{U}'$ must satisfy the Navier–Stokes equations,

$$\mu\nabla^2\mathbf{U}' - \nabla p' = \rho\mathbf{U}' \cdot \nabla\mathbf{U}',$$

$$\nabla \cdot \mathbf{U}' = 0, \qquad [2.4]$$

it is seen by substituting from [2.1] that $U_2'(r_1')$ must be a quadratic function of r_1' so that

$$\mathbf{U}'(\mathbf{r}') = (a^* + b^*r_1' + c^*r_2'^2)\mathbf{e}_2, \qquad [2.5]$$

$\mathbf{e}_2$ being a unit vector along the r_2' direction (with $\mathbf{e}_1$ and $\mathbf{e}_3$ similarly defined). The no slip boundary condition to be satisfied on the wall at $r_1' = 0$ then requires the wall velocity $\mathbf{U}_w'$ to be

$$\mathbf{U}_w' = a^*\mathbf{e}_2. \qquad [2.6]$$

We use throughout (unless otherwise stated) dimensionless unprimed variables based upon the velocity V, length a and fluid viscosity μ so that the dimensionless velocity $\mathbf{U} = \mathbf{U}'/V$ and the dimensionless position vector $\mathbf{r} = \mathbf{r}'/a$.

A condition that the theory described by Cox & Brenner (1968) apply is that there is either no outer inertial expansion or that, when there is an outer inertial expansion it gives rise to no contribution to the force on the particle to the order considered (i.e. to order Re^{+1}). It was shown that sufficient conditions for this to occur are

$$Re \ll \kappa \ll 1, \qquad [2.7]$$

and also that

$$|\mathbf{U}|\,|\mathbf{v}_0^*| = 0(r^{-1-\alpha}) \quad \text{as} \quad r \to \infty \quad \text{for} \quad \alpha > 0, \qquad [2.8]$$

where $\mathbf{v}_0^*$ is the disturbance flow produced by the sphere and wall calculated on the basis of the creeping motion equations and $r = |\mathbf{r}|$. Since it was shown by Oseen (1927) that $\mathbf{v}_0^*$ is $0(r^{-2})$ as $r \to \infty$ (see also [4.31]) for the present case, the condition [2.8] reduces to

$$|\mathbf{U}| = 0(r^{1-\alpha}) \quad \text{as} \quad r \to \infty.$$

Thus it would appear that the results could not be applied to the present problem for which $\mathbf{U} = 0(r^2)$ as $r \to \infty$. However it will now be shown that in the present case the condition [2.8] may be relaxed somewhat if one is concerned only with finding the velocity of migration for the sphere towards or away from the wall to order Re^{+1}.

Taking our axes to be moving in the r_2 direction with the sphere so that the motion is steady (and neglecting unsteadiness resulting from the small migration velocity) the dimensionless fluid velocity $\mathbf{v}$ satisfies

$$\begin{aligned} \nabla^2\mathbf{v} - \nabla p &= Re\,\mathbf{v}\cdot\nabla\mathbf{v} \\ \nabla\cdot\mathbf{v} &= 0 \\ \mathbf{v} &= 0 \quad \text{on } W, \end{aligned} \qquad [2.9]$$

and also the no slip boundary condition on the sphere surface.

Considering the parameter κ fixed and making an expansion in Re, an inner inertial expansion is obtained using $\mathbf{r}$ as an independent variable. This expansion will be shown to be

$$\mathbf{v} = (\mathbf{U} + \mathbf{v}_0^*) + Re\,\mathbf{v}_1^* + \cdots, \qquad [2.10]$$

where $\mathbf{v}_0^*$ is the creeping motion disturbance velocity produced by the sphere and wall mentioned previously and as such cannot give rise to migration of the sphere. Thus $\mathbf{v}_0^*$ is of order r^{-2} as $r \to \infty$.

$\mathbf{v}_1^*$ then satisfies

$$\begin{aligned} \nabla^2\mathbf{v}_1^* - \nabla p_1^* &= (\mathbf{U}\cdot\nabla\mathbf{v}_0^* + \mathbf{v}_0^*\cdot\nabla\mathbf{U} + \mathbf{v}_0^*\cdot\nabla\mathbf{v}_0^*), \\ \nabla\cdot\mathbf{v}_1^* &= 0, \\ \mathbf{v}_1^* &= \mathbf{0} \quad \text{on } W, \end{aligned} \qquad [2.11]$$

and $\mathbf{v}_1^* = \mathbf{0}$ on the sphere $r = 1$, with boundary conditions at infinity to be obtained by matching onto an outer expansion.

One may write

$$\mathbf{v}_1^* = (\mathbf{v}_1^*)_{c.F.} + (\mathbf{v}_1^*)_{p.I} \qquad [2.12]$$

as the sum of a complementary function and particular integral of [2.11].

Thus $(\mathbf{v}_1^*)_{p.I.}$ is any particular solution of [2.11] while $(\mathbf{v}_1^*)_{c.F.}$ is a solution of the creeping motion equations (which may tend to infinity as $r \to \infty$) with $(\mathbf{v}_1^*)_{c.F.} = \mathbf{0}$ on the wall W and on the sphere.

From the form of $\mathbf{U}$ and $\mathbf{v}_0^*$ as $r \to \infty$ it is observed that $(\mathbf{v}_1^*)_{P.I.}$ possesses terms of order r^1, r^0, $r^{-1}, \ldots$ as $r \to \infty$. Defining an outer inertial expansion with independent variable $\bar{\mathbf{r}}$ such that

$$\bar{\mathbf{r}} = Re^{1/3}\mathbf{r}, \qquad [2.13]$$

the terms in $\mathbf{v}_0^*$ then match terms of order $Re^{2/3}\bar{r}^{-2}$, $Re\bar{r}^{-3}, \ldots$ in the outer expansion as $\bar{r} \to 0$. In the outer expansion, one therefore has

$$\mathbf{v} = \mathbf{U} + Re^{2/3}\bar{\mathbf{u}}_1 + Re\bar{\mathbf{u}}_2 + \cdots. \qquad [2.14]$$

The equations for $\bar{\mathbf{u}}_1$ are

$$\nabla^2\bar{\mathbf{u}}_1 - \nabla\bar{p}_1 = c^*\bar{r}_1{}^2\frac{\partial}{\partial\bar{r}_2}\bar{\mathbf{u}}_1 + 2c^*\bar{r}_1(\bar{u}_1)_1\,\mathbf{e}_2,$$

$$\nabla\cdot\bar{\mathbf{u}}_1 = 0,$$

$$\bar{\mathbf{u}}_1 = \mathbf{0} \quad \text{on} \quad W,$$

and

$$\bar{\mathbf{u}}_1 \to 0 \quad \text{as} \quad \bar{r} \to \infty, \qquad [2.15]$$

and for $\bar{\mathbf{u}}_2$ are

$$\nabla^2\bar{\mathbf{u}}_2 - \nabla\bar{p}_2 = c^*\bar{r}_1{}^2\frac{\partial}{\partial\bar{r}_2}\bar{\mathbf{u}}_2 + 2c^*\bar{r}_1(\bar{u}_2)_1\,\mathbf{e}_2$$

$$+ b^*\bar{r}_1\frac{\partial}{\partial\bar{r}_2}\bar{\mathbf{u}}_1 + b^*(\bar{u}_1)_1\mathbf{e}_2,$$

$$\nabla\cdot\bar{\mathbf{u}}_2 = 0,$$

$$\bar{\mathbf{u}}_2 = \mathbf{0} \quad \text{on} \quad W,$$

and

$$\bar{\mathbf{u}}_2 \to 0 \quad \text{as} \quad \bar{r} \to \infty, \qquad [2.16]$$

all differentiation being taken with respect to the outer $\bar{\mathbf{r}}$ variable. The inner boundary conditions on $\bar{\mathbf{u}}_1$ and $\bar{\mathbf{u}}_2$ are obtained by matching the inner expansion. The form of the part $(\mathbf{v}_1^*)_{P.I.}$ of $\mathbf{v}_1^*$ in the inner expansion for large r matches terms of order $Re^{2/3}\bar{r}^1, Re\bar{r}^0, Re^{4/3}\bar{r}^{-1}, \ldots$ in the outer expansion for $\bar{r} \to 0$. Thus $\bar{\mathbf{u}}_1$ must contain at least terms like $\bar{r}^{-2}$, $\bar{r}^{+1}$ while $\bar{\mathbf{u}}_2$ must contain terms like $\bar{r}^{-3}$, $\bar{r}^0$ for $\bar{r} \to 0$. No terms like $\bar{r}^{-1}$ or $\bar{r}^0$ can occur in the symptotic form of $\bar{\mathbf{u}}_1$ for $\bar{r} \to 0$ since otherwise they would match terms proportional to $Re^{1/3}$ and $Re^{2/3}$ respectively in the inner expansion. Such terms cannot occur since they would have to satisfy the creeping flow equations with zero velocity on wall W and on sphere and be of order r^{-1} and r^0 respectively as $r \to \infty$, there being no non-trivial solutions for these problems. This can be proved for the former problem by noting that no work is done on the fluid and for the latter problem by noting that for large r, the solution which is homogenous in r^0 is a uniform flow which has to be identically zero for zero velocity on W.

Similarly it may be shown that $\bar{\mathbf{u}}_2$ contains no terms like $\bar{r}^{-2}$ or $\bar{r}^{-1}$ in its asymptotic expansion for $\bar{r} \to 0$. Thus $\bar{\mathbf{u}}_1$ contains terms $\bar{r}_1{}^{-2}\bar{r}_1{}^{+1}\ldots$ and $\bar{\mathbf{u}}_2$ terms like $\bar{r}_1{}^{-3}\bar{r}_1{}^0\ldots$ in their asymptotic expansion for $\bar{r} \to 0$, while the form of the inner expansion is indeed that given by [2.10].

Consider now the term $(\mathbf{v}_1^*)_{C.F.}$ in the inner expansion which satisfies the creeping flow equations with $(\mathbf{v}_1^*)_{c.F.} = \mathbf{0}$ on W and on the sphere and whose outer boundary conditions is to be determined by matching the outer expansion.

If $\mathbf{v}_1^*$ contains a term like r^n as $r\rightarrow\infty$ then this matches a term order $Re^{(3-n)/3}\bar{r}^n$ in the outer expansion. However, from the form of the outer expansion given in [2.14], it is seen that

$$(3-n)/3 \geqslant 2/3 \quad \text{or} \quad n \leqslant 1.$$

Thus the asymptotic form of $(\mathbf{v}_1^*)_{c.F.}$ contains terms of order $r_1{}^1 r_1{}^0\ldots$ and since each term satisfies the creeping motion equation,

$$[(\mathbf{v}_1^*)_{c.F.}]_i = A_{ij}r_j + B_i + 0(r^{-1}) \quad \text{as} \quad r\rightarrow\infty, \qquad [2.17]$$

where $A_{ii} = 0$ from the continuity equation. This velocity $(\mathbf{v}_1^*)_{c.F.}$ must be zero on $r_1 = 0$ for all r_2 and r_3. This implies

$$B_i = 0 \quad \text{and} \quad A_{i2} = A_{i3} = 0 \quad \text{for all} \quad i.$$

Also since $A_{ii} = 0$ one has $A_{11} = 0$.
Thus as $r\rightarrow\infty$

$$[(\mathbf{v}_1^*)_{c.F.}]_i = A_{21}\delta_{i2}r_1 + A_{31}r_1\delta_{i3} + 0(r^{-1})$$

which represents a plane shear flow. The flow field $(\mathbf{v}_1^*)_{c.F.}$ cannot therefore give rise to any force on the sphere in the r_1 direction. Thus in calculating the inertial migration towards or away from the wall, one does not need to calculate $(\mathbf{v}_1^*)_{c.F.}$ and so the entire calculation for such migration may be done without reference to the outer expansion.

The results derived by Cox & Brenner (1968) may therefore be used without modification for an undisturbed flow field given by [2.5] even though the condition [2.8] is not satisfied. This calculation was done by expanding the velocity fields $\mathbf{v}_0^*$ and $\mathbf{v}_1^*$ in the inner inertial expansion (see [2.10]) in a power series in κ. This required also an inner and an outer expansion, the inner expansion using the dimensionless independent variable $\mathbf{r}$ while the outer expansion used $\bar{\mathbf{r}}$ where

$$\bar{\mathbf{r}} = \kappa\mathbf{r} = \mathbf{r}'/d$$

was the position variable made dimensionless with respect to d. It was shown that the particle migration resulted from inertia effects in the outer expansion, so that the existence of the wall and the precise details of the undisturbed flow even at large distances are important.

Letting V'_∞ be the dimensional velocity with which the sphere would move in the r_2 direction in a quiescent unbounded fluid as the result of sedimentation and buoyancy, the results obtained by Cox & Brenner (1968) for three different situations may be expressed as follows:

1. A quiescent or nearly quiescent fluid for which the condition $|V'_\infty/V| \gg 1$ is satisfied, the dimensional migration velocity $u_M{}^{(1)}(d)$ of the sphere at a distance d from the wall in the r_1 direction being obtained as

$$u_M{}^{(1)}(d) = \frac{6\pi a V'^2_\infty}{\nu} h. \qquad [2.18]$$

2. A non-neutrally buoyant sphere for which $\kappa^2 \ll |V'_\infty/V| \ll 1$, the migration velocity $u_M{}^{(2)}(d)$ being obtained as

$$u_M{}^{(2)}(d) = -\frac{6\pi a V'_\infty V}{\nu} g. \qquad [2.19]$$

3. A neutrally or almost neutrally buoyant sphere for which $|V'_\infty/V| \ll \kappa^2$, the migration velocity $u_M^{(3)}(d)$ being obtained as

$$u_M^{(3)}(d) = \frac{10}{3}\pi\kappa^2 \frac{aV^2}{\nu}(f_1 + f_2) \qquad [2.20]$$

for a sphere free to rotate, and

$$u_M^{(3)}(d) = \frac{4}{3}\pi\kappa^2 \frac{aV^2}{\nu}(4f_1 + f_2) \qquad [2.21]$$

for a sphere prevented from rotating.

The results given by [2.18] and [2.19] are valid whether or not the sphere is free to rotate. The quantities h, g, f_1 and f_2 were obtained as volume integrals over the region $\bar{r}_1 > 0$ (denoted by Γ) in the form

$$h = \int_\Gamma V_{i1} \frac{\partial}{\partial \bar{r}_2} V_{i2} d\bar{\mathbf{r}}, \qquad [2.22]$$

$$g = \int_\Gamma \left\{ [U_2(\bar{r}_1) - U_2(\bar{r}_1^*)] V_{i1} \frac{\partial}{\partial \bar{r}_2} V_{i2} + \frac{\partial U_2(\bar{\mathbf{r}})}{\partial \bar{r}_i} V_{i2} V_{21} \right\} d\bar{\mathbf{r}}, \qquad [2.23]$$

$$f_1 = \left[\frac{\partial U_2}{\partial \bar{r}_1}\right]_{\bar{r}_1 = 1} \int_\Gamma \left\{ [U_2(\bar{r}_1) - U_2(\bar{r}_1^*)] V_{i1} \frac{\partial^2}{\partial \bar{r}_1^* \partial \bar{r}_2} V_{i2} + V_{21} \frac{\partial U_2}{\partial \bar{r}_1} \frac{\partial V_{12}}{\partial \bar{r}_1^*} \right\} d\bar{\mathbf{r}}, \qquad [2.24]$$

$$f_2 = \left[\frac{\partial U_2}{\partial \bar{r}_1}\right]_{\bar{r}_1 = 1} \int_\Gamma \left\{ [U_2(\bar{r}_1) - U_2(\bar{r}_1^*)] V_{i1} \frac{\partial^2}{\partial \bar{r}_2^* \partial \bar{r}_2} V_{i1} + V_{21} \frac{\partial U_2}{\partial \bar{r}_1} \frac{\partial V_{11}}{\partial \bar{r}_2^*} \right\} d\bar{\mathbf{r}}, \qquad [2.25]$$

where $\bar{\mathbf{r}}^*$ is the dimensionless position vector of the sphere [and is thus (1, 0, 0)], and the quantity $V_{ij}(\bar{\mathbf{r}}, \bar{\mathbf{r}}^*)$ is the Green's function for creeping flow in the presence of the wall W and satisfies

$$\frac{\partial^2 V_{ij}}{\partial \bar{r}_k \partial \bar{r}_k} - \frac{\partial P_j}{\partial \bar{r}_i} + \delta_{ij}\delta(\bar{\mathbf{r}} - \bar{\mathbf{r}}^*) = 0, \quad \frac{\partial V_{ij}}{\partial r_i} = 0,$$

with

$$V_{ij} = 0 \quad \text{on} \quad \bar{r}_1 = 0,$$
$$V_{ij} \to 0 \quad \text{as} \quad \bar{r} \to \infty. \qquad [2.26]$$

Since these equations and boundary conditions for V_{ij} are linear, it follows that the creeping flow produced by a point force $\mathbf{F}$ acting at $\bar{\mathbf{r}}^*$ and satisfying the no slip boundary condition on $\bar{r}_1 = 0$ is $\mathbf{u}(\bar{\mathbf{r}})$, where

$$u_i(\bar{\mathbf{r}}) = V_{ij}(\bar{\mathbf{r}}, \bar{\mathbf{r}}^*) F_j. \qquad [2.27]$$

The dimensionless flow field $U_2(\bar{r}_1)$ from [2.5] and [2.6] is given by

$$U_2 - (U_W)_2 = (b^* r_1' + c^* r_1'^2)/V,$$

where $(U_W)_2$ is the value of U_2 on the wall $r_1' = 0$. If this velocity field attains its maximum value at $r_1' = 1$ (as shown in figure 1) and if V is chosen to be the value of U_2' at $r_1 = l$, then

$$U_2 - (U_W)_2 = 2\beta\bar{r}_1 - \beta^2\bar{r}_1^2, \qquad [2.28]$$

where $\beta = d/l$.

Thus in [2.22]–[2.25]

$$[U_2(\tilde{r}_1) - U_2(\tilde{r}_1^*)] = 2\beta(\tilde{r}_1 - \tilde{r}_1^*) - \beta^2(\tilde{r}_1^2 - \tilde{r}_1^{*2}), \quad [2.29]$$

$$\frac{\partial U_2}{\partial \tilde{r}_1} = 2\beta - 2\beta^2 \tilde{r}_1. \quad [2.30]$$

3. A POINT FORCE NEAR A PLANE WALL

In order to evaluate the integrals [2.22]–[2.25] determining the migration velocity, the value of the Green's function $V_{ij}(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}^*)$ satisfying [2.26] must be obtained. In order to do this consider the creeping motion flow $\mathbf{u}$ (and pressure p) produced by a point force of strength $\mathbf{F}$ acting at $\tilde{\mathbf{r}}^*$ in the presence of the plane rigid wall at $\tilde{r}_1 = 0$ [see figure 2]. Then

$$\tilde{\nabla}^2 \mathbf{u} - \tilde{\nabla} p = \mathbf{0}, \qquad \tilde{\nabla} \cdot \mathbf{u} = 0, \quad [3.1]$$

with boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \tilde{r}_1 = 0, \quad [3.2]$$

$$\mathbf{u} \sim \frac{1}{8\pi}\left[\frac{\mathbf{F}}{|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^*|} + \frac{(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^*)(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^*)\cdot \mathbf{F}}{|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^*|^3}\right], \qquad \tilde{\mathbf{r}} \to \tilde{\mathbf{r}}^*. \quad [3.3]$$

Let $\bar{\mathbf{u}}$ be the velocity and $\bar{p}$ the pressure of a creeping motion flow produced by a surface distribution of forces $\mathbf{f}(\tilde{\mathbf{r}}_s)$ on the $\tilde{r}_1 = 0$ plane, where $\tilde{\mathbf{r}}_s = (0, \tilde{r}_2, \tilde{r}_3)$ is a vector lying in the surface $\tilde{r}_1 = 0$. Then $\bar{\mathbf{u}}$ and $\bar{p}$ are given by

$$\bar{\mathbf{u}}(\tilde{\mathbf{r}}) = \frac{1}{8\pi}\int \left[\frac{\mathbf{I}}{|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}_s'|} + \frac{(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}_s')(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}_s')}{|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}_s'|}\right]\cdot \mathbf{f}(\tilde{\mathbf{r}}_s')\, d\tilde{\mathbf{r}}_s',$$

$$\bar{p}(\tilde{\mathbf{r}}) = \frac{1}{4\pi}\int \frac{\tilde{\mathbf{r}} - \tilde{\mathbf{r}}_s'}{|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}_s'|^3}\cdot \mathbf{f}(\tilde{\mathbf{r}}_s')\, d\tilde{\mathbf{r}}_s', \quad [3.4]$$

the integration being taken over the plane $\tilde{r}_1 = 0$.

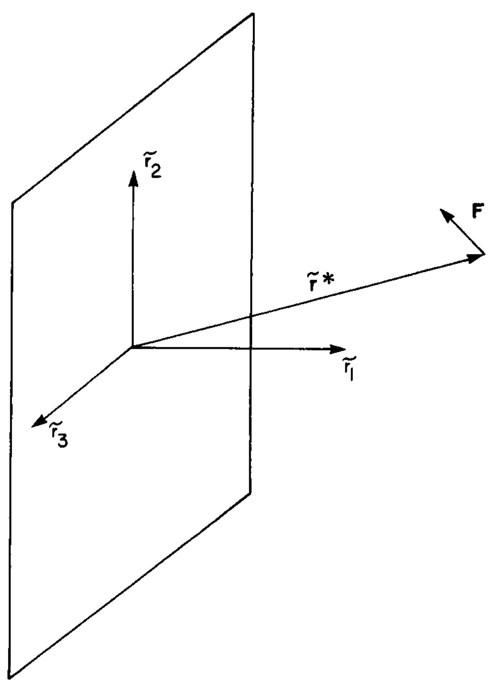

Figure 2. Point force $\mathbf{F}$ acting at position $\tilde{\mathbf{r}}^*$ in the neighbourhood of the plane wall $\tilde{r}_1 = 0$.

The stress tensor $\bar{\mathbf{P}}$ (with components $\bar{P}_{ij}$) for this flow field $(\bar{\mathbf{u}}, \bar{p})$ is obtained from [3.4] as

$$\bar{\mathbf{P}} = -\frac{3}{4\pi}\int \frac{(\bar{\mathbf{r}}-\bar{\mathbf{r}}'_s)(\bar{\mathbf{r}}-\bar{\mathbf{r}}'_s)(\bar{\mathbf{r}}-\bar{\mathbf{r}}'_s)}{|\bar{\mathbf{r}}-\bar{\mathbf{r}}'_s|^5}\cdot \mathbf{f}(\bar{\mathbf{r}}'_s)\,d\bar{\mathbf{r}}'_s. \qquad [3.5]$$

In the limit $\bar{r}_1 \to 0$, the contribution to

$$\bar{P}_{i1} = -\frac{3}{4\pi}\bar{r}_1\int \frac{(\bar{\mathbf{r}}-\bar{\mathbf{r}}'_s)(\bar{\mathbf{r}}-\bar{\mathbf{r}}'_s)\cdot \mathbf{f}(\bar{\mathbf{r}}'_s)}{|\bar{\mathbf{r}}-\bar{\mathbf{r}}'_s|^5}\,d\bar{\mathbf{r}}'_s \qquad [3.6]$$

from integration in the region outside of a circle of fixed radius ϵ ($\epsilon \ll 1$) centred on $\bar{\mathbf{r}}_s = (0, \bar{r}_2, \bar{r}_3)$ is negligible so that one may replace $\mathbf{f}(\bar{\mathbf{r}}_s')$ by $\mathbf{f}(\bar{\mathbf{r}}_s)$ in the integrand, yielding

$$\bar{P}_{i1} = -\frac{3}{4\pi}\bar{r}_1\mathbf{f}(\bar{\mathbf{r}}_s)\cdot\int \frac{(\bar{\mathbf{r}}-\bar{\mathbf{r}}'_s)(\bar{\mathbf{r}}-\bar{\mathbf{r}}'_s)}{|\bar{\mathbf{r}}-\bar{\mathbf{r}}'_s|^5}\,d\bar{\mathbf{r}}'_s. \qquad [3.7]$$

It may readily be shown that

$$\int \frac{(\bar{\mathbf{r}}-\bar{\mathbf{r}}'_s)(\bar{\mathbf{r}}-\bar{\mathbf{r}}'_s)}{|\bar{\mathbf{r}}-\bar{\mathbf{r}}'_s|^5}\,d\bar{\mathbf{r}}'_s = \frac{2\pi}{3|r_1|}\mathbf{I},$$

where $\mathbf{I}$ is the unit tensor. Thus as $\bar{r}_1 \to 0$

$$\begin{aligned} \bar{P}_{i1} &\to -\frac{1}{2}f_i(\bar{\mathbf{r}}_s) \quad \text{for} \quad \bar{r}_1 > 0, \\ &\to +\frac{1}{2}f_i(\bar{\mathbf{r}}_s) \quad \text{for} \quad \bar{r}_1 < 0, \end{aligned} \qquad [3.8]$$

giving the values of $\bar{P}_{i1}$ on either side of the plane $\bar{r}_1 = 0$.

Define a velocity $\mathbf{u}^*$ and pressure p^* as that produced by a point force $\mathbf{F}$ acting at the point $\bar{\mathbf{r}} = \bar{\mathbf{r}}^*$, the fluid now being considered unbounded. Then near $\bar{\mathbf{r}} = \bar{\mathbf{r}}^*$ the velocity $\mathbf{u}^*$ possesses a singularity of the type given by [3.3].

The flow field $(\bar{\mathbf{u}}, \bar{p})$ considered above is chosen as the flow produced by surface forces of strength $\mathbf{f}(\bar{\mathbf{r}})_s$ acting on $\bar{r}_1 = 0$, where

$$\mathbf{f}(\bar{\mathbf{r}}_s) = -2P^*_{i1}(\bar{\mathbf{r}}_s), \qquad [3.9]$$

the quantity $P^*_{ij}(\bar{\mathbf{r}}_s)$ being the value of the stress tensor $\mathbf{P}^*$ of the flow $(\mathbf{u}^*, p^*)$ evaluated at the position $\bar{\mathbf{r}}_s$ on the plane $\bar{r}_1 = 0$. Then by [3.8], it is seen that on the two sides of $\bar{r}_1 = 0$ one has

$$\begin{aligned} \bar{P}_{i1} &\to P^*_{i1}(\bar{\mathbf{r}}_s) \quad \text{as} \quad \bar{r}_1 \to 0 \quad \text{with} \quad \bar{r}_1 > 0, \\ &\to -P^*_{i1}(\bar{\mathbf{r}}_s) \quad \text{as} \quad \bar{r}_1 \to 0 \quad \text{with} \quad \bar{r}_1 < 0. \end{aligned} \qquad [3.10]$$

Defining a velocity field $\hat{\mathbf{u}}$ and pressure $\hat{p}$ as

$$\begin{aligned} \hat{\mathbf{u}} &= \bar{\mathbf{u}} + \mathbf{u}^*, \\ \hat{p} &= \bar{p} + p^*, \end{aligned} \qquad [3.11]$$

it follows that $(\hat{\mathbf{u}}, \hat{p})$ like the flow fields $(\bar{\mathbf{u}}, \bar{p})$ and $(\mathbf{u}^*, p^*)$ must satisfy the creeping motion equations [3.1]. Also since $(\bar{\mathbf{u}}, \bar{p})$ contains no singularity and $(\mathbf{u}^*, p^*)$ is of the form given by [3.3] near $\bar{\mathbf{r}} = \bar{\mathbf{r}}^*$ it follows that $\hat{\mathbf{u}}$ has a singularity of the form [3.3].

If $\hat{\mathbf{P}}$ is the stress tensor corresponding to the flow field $(\hat{\mathbf{u}}, \hat{p})$ then by [3.10], it is seen that on side $\bar{r}_1 > 0$ of the plane $\bar{r}_1 = 0$,

$$\hat{P}_{i1} \to 2P^*_{i1}(\bar{\mathbf{r}}_s) \quad \text{as} \quad \bar{r}_1 \to 0, \qquad [3.12]$$

whilst on the side $\bar{r}_1 < 0$,

$$\hat{P}_{i1} \to 0 \quad \text{as} \quad \bar{r}_1 \to 0. \qquad [3.13]$$

Consider the volume $\bar{r}_1 < 0$ of fluid undergoing the flow $(\hat{\mathbf{u}}, \hat{p})$. The work done on this fluid by the plane $\bar{r}_1 = 0$ is $-\int \hat{u}_i \hat{P}_{i1}\, d\bar{r}_2\, d\bar{r}_3$ integrated over the plane and hence is zero by [3.13]. Also since $\bar{\mathbf{u}}$ and $\mathbf{u}^*$ (and hence $\hat{\mathbf{u}}$) are of order $\bar{r}^{-1}$ as $\bar{r} \to \infty$ it follows that no work is done on the fluid at infinity. Hence, no work is done in order to produce the flow $(\hat{\mathbf{u}}, \hat{p})$ in this volume $\bar{r}_1 < 0$. Therefore there can be no viscous dissipation due to $(\hat{\mathbf{u}}, p)$ in $\bar{r}_1 < 0$ and so it must represent a rigid body motion there. However, since $\hat{\mathbf{u}} = 0\,(\bar{r}^{-1})$ as $\bar{r} \to \infty$ it follows that $\hat{\mathbf{u}} = \mathbf{0}$ everywhere in the volume $\bar{r}_1 < 0$. Also since $\bar{\mathbf{u}}$ and $\mathbf{u}^*$ (and hence $\hat{\mathbf{u}}$) are continuous across the plane $\bar{r}_1 = 0$ it is seen that $\hat{\mathbf{u}} = \mathbf{0}$ on $\bar{r}_1 = 0$. Therefore the flow field $(\hat{\mathbf{u}}, \hat{p})$ satisfies the same equations [3.1], [3.2] and [3.3] as the flow field $(\mathbf{u}, p)$ in the volume $\bar{r}_1 \geq 0$. Since the solution of such equations must be unique, it follows that $\hat{\mathbf{u}}$ is given by

$$\hat{\mathbf{u}} = \mathbf{0} \quad \text{for} \quad \bar{r}_1 < 0,$$
$$\hat{\mathbf{u}} = \mathbf{u} \quad \text{for} \quad \bar{r}_1 > 0.$$

Thus it is seen that the solution of [3.1], [3.2] and [3.3] for $\mathbf{u}$ is represented by the flow field produced by a point force $\mathbf{F}$ at $\bar{\mathbf{r}} = \bar{\mathbf{r}}^*$ together with a surface distribution of forces $-2P^*_{i1}(\bar{\mathbf{r}}_s)$ on $\bar{r}_1 = 0$. Such a flow field is then defined not only in the space $\bar{r}_1 \geq 0$, but also in the space $\bar{r}_1 < 0$ where it represents the flow field $\mathbf{u} = \mathbf{0}$.

4. FOURIER TRANSFORM OF FLOWS

It will be found convenient for the evaluation of the integrals [2.22] to [2.25] to obtain the Fourier transform of the flow fields $(\mathbf{u}^*, p^*)$, $(\bar{\mathbf{u}}, \bar{p})$ and hence $(\hat{\mathbf{u}}, \hat{p})$. Thus we define $\mathbf{\Gamma}^*$ and Π^* as the three dimensional Fourier transforms of the velocity $\mathbf{u}^*$ and pressure p^* respectively, so that

$$\mathbf{\Gamma}^*(\mathbf{K}) = \int \mathbf{u}^*(\bar{\mathbf{r}}) \exp(i\mathbf{K} \cdot \bar{\mathbf{r}})\, d\bar{\mathbf{r}},$$
$$\Pi^*(\mathbf{K}) = \int p^*(\bar{\mathbf{r}}) \exp(i\mathbf{K} \cdot \bar{\mathbf{r}})\, d\bar{\mathbf{r}}, \qquad [4.1]$$

the integrals being taken over the whole of space. $\mathbf{u}^*$ and p^* are then given by the inverse Fourier transforms

$$\mathbf{u}^*(\bar{\mathbf{r}}) = \frac{1}{8\pi^3} \int \mathbf{\Gamma}^*(\mathbf{K}) \exp(-i\mathbf{K} \cdot \bar{\mathbf{r}})\, d\mathbf{K},$$
$$p^*(\bar{\mathbf{r}}) = \frac{1}{8\pi^3} \int \Pi^*(\mathbf{K}) \exp(-i\mathbf{K} \cdot \bar{\mathbf{r}})\, d\mathbf{K}, \qquad [4.2]$$

these integrations being taken over the whole of $\mathbf{K}$ space.
$(\bar{\mathbf{\Gamma}}, \bar{\Pi})$ and $(\hat{\mathbf{\Gamma}}, \hat{\Pi})$ are similarly defined as the three dimensional Fourier transforms of $(\bar{\mathbf{u}}, \bar{p})$ and $(\hat{\mathbf{r}}, \hat{p})$ respectively.
Since the flow field $(\bar{\mathbf{u}}, p^*)$ satisfies

$$\bar{\nabla}^2 \mathbf{u}^* - \bar{\nabla} p^* + \mathbf{F}\delta(\bar{\mathbf{r}} - \bar{\mathbf{r}}^*) = \mathbf{0}, \qquad [4.3]$$
$$\bar{\nabla} \cdot \mathbf{u}^* = 0,$$

with

$$\mathbf{u}^* \to 0 \quad \text{as} \quad \tilde{r} \to \infty, \qquad [4.4]$$

where $\delta(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^*)$ is the three dimensional Dirac delta function, one obtains the equations for $(\mathbf{\Gamma}^*, \Pi^*)$ by taking the Fourier transform of [4.3] as

$$-K^2\mathbf{\Gamma}^* + i\Pi^*\mathbf{K} + \mathbf{F}\exp(i\mathbf{K}\cdot\tilde{\mathbf{r}}^*) = 0, \qquad [4.5a]$$

$$\mathbf{K}\cdot\mathbf{\Gamma}^* = 0. \qquad [4.5b]$$

These equations possess the solution

$$\mathbf{\Gamma}^*(\mathbf{K}) = \frac{K^2\mathbf{I} - \mathbf{K}\mathbf{K}}{K^4}\cdot\mathbf{F}\exp(i\mathbf{K}\cdot\tilde{\mathbf{r}}^*), \qquad [4.6a]$$

$$\Pi^*(\mathbf{K}) = \frac{i\mathbf{K}\cdot\mathbf{F}}{K^2}\exp(i\mathbf{K}\cdot\tilde{\mathbf{r}}^*), \qquad [4.6b]$$

where $K^2 = |\mathbf{K}|^2 = K_1^2 + K_2^2 + K_3^2$. Hence, noting that this solution $(\mathbf{\Gamma}^*, \Pi^*)$ would give $\mathbf{u}^*$ which automatically tends to zero as $\tilde{r} \to \infty$, it follows that these expressions [4.6] give the required Fourier transform of $(\mathbf{u}^*, p^*)$. Since the stress tensor P^*_{ij} is defined as

$$P^*_{ij} = -p^*\delta_{ij} + \frac{\partial u^*_i}{\partial \tilde{r}_j} + \frac{\partial u^*_j}{\partial \tilde{r}_i}, \qquad [4.7]$$

its Fourier transform $\mathbf{\Pi}^*$ is

$$\mathbf{\Pi}^* = -\Pi^*\mathbf{I} + i(\mathbf{\Gamma}^*\mathbf{K} + \mathbf{K}\mathbf{\Gamma}^*), \qquad [4.8]$$

which, upon substitution from [4.6], yields

$$\mathbf{\Pi}^* = -i\mathbf{F}\cdot\mathbf{T}^{(3)}(\mathbf{K})\exp(i\mathbf{K}\cdot\tilde{\mathbf{r}}^*), \qquad [4.9]$$

where $\mathbf{T}^{(3)}(\mathbf{K})$ is the third rank symmetric tensor,

$$T^{(3)}_{ijk}(\mathbf{K}) = \frac{\delta_{ij}K_k + \delta_{ik}K_j + \delta_{jk}K_i}{K^2} - 2\frac{K_iK_jK_k}{K^4}. \qquad [4.10]$$

The inverse transform then gives

$$\mathbf{P}^* = -\frac{i}{8\pi^3}\int \mathbf{F}\cdot\mathbf{T}^{(3)}(\mathbf{K})\exp\{-i\mathbf{K}\cdot(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^*)\}\,d\mathbf{K}, \qquad [4.11]$$

so that the force distribution $-2P^*_{i1}(\tilde{\mathbf{r}}_s)$ on the plane $\tilde{r}_1 = 0$ may be written as

$$-2P^*_{i1}(\tilde{\mathbf{r}}_s) = +\frac{i}{4\pi^3}\int \mathbf{F}\cdot\mathbf{T}^{(3)}(\mathbf{K})\cdot\mathbf{e}_1\exp\{-i\mathbf{K}\cdot(\tilde{\mathbf{r}}_s - \tilde{\mathbf{r}}^*)\}\,d\mathbf{K}, \qquad [4.12]$$

where $\mathbf{e}_1$ is a unit vector in the $\tilde{r}_1$-direction ($\mathbf{e}_2$ and $\mathbf{e}_3$ being similarly defined).

The flow field $(\bar{\mathbf{u}}, \bar{p})$ produced by this force distribution satisfies

$$\tilde{\nabla}^2\bar{\mathbf{u}} - \tilde{\nabla}\bar{p} + \mathscr{F}(\tilde{\mathbf{r}}_s)\delta(\tilde{r}_1) = 0,$$

$$\nabla\cdot\bar{\mathbf{u}} = 0, \qquad [4.13]$$

with

$$\bar{\mathbf{u}} \to 0 \quad \text{as} \quad \bar{r} \to \infty, \tag{4.14}$$

where $\mathscr{F}_i(\bar{\mathbf{r}}_s) = -2P^*_{i1}(\bar{\mathbf{r}}_s)$. Proceeding as before, one may obtain equations for $\bar{\Gamma}$ and $\bar{\Pi}$ as (see [4.5])

$$-K^2\bar{\Gamma} + i\bar{\Pi}\mathbf{K} + \int \mathscr{F}(\bar{\mathbf{r}}_s)\exp(i\mathbf{K}_s \cdot \bar{\mathbf{r}}_s)\,\mathrm{d}\bar{\mathbf{r}}_s = 0,$$

$$\mathbf{K} \cdot \bar{\Gamma} = 0, \tag{4.15}$$

where $(\mathbf{K}_s)_i = K_2\delta_{i2} + K_3\delta_{i3}$ is a vector lying in the plane $K_1 = 0$. Thus the values of $\bar{\Gamma}$ and $\bar{\Pi}$ are obtained as

$$\bar{\Gamma} = \frac{K^2\mathbf{I} - \mathbf{KK}}{K^4} \cdot \int \mathscr{F}(\bar{\mathbf{r}}_s)\, exp\, (i\mathbf{K}_s \cdot \bar{\mathbf{r}}_s)\,\mathrm{d}\bar{\mathbf{r}}_s,$$

$$\bar{\Pi} = \frac{i\mathbf{K}}{K^2} \cdot \int \mathscr{F}(\bar{\mathbf{r}}_s)\exp(i\mathbf{K} \cdot \bar{\mathbf{r}}_s)\, d\bar{r}_s. \tag{4.16}$$

Taking the double Fourier transform (with respect to $\bar{r}_2$ and $\bar{r}_3$) of the expression for $\mathscr{F}(\bar{\mathbf{r}}_s)$ given by [4.12], one obtains

$$\int \mathscr{F}(\bar{\mathbf{r}}_s)\exp(i\mathbf{K}_s \cdot \bar{\mathbf{r}}_s)\,\mathrm{d}\bar{\mathbf{r}}_s = \frac{i}{\pi}\int \mathbf{F} \cdot \mathbf{T}^{(3)}(\mathbf{K}) \cdot \mathbf{e}_1 \exp(i\mathbf{K} \cdot \bar{\mathbf{r}}^*)\,\mathrm{d}K_1$$

$$= \frac{i}{\pi}\exp(i\mathbf{K}_s \cdot \bar{\mathbf{r}}^*_s)\mathbf{F} \cdot \int \mathbf{T}^{(3)}(\mathbf{K}) \cdot \mathbf{e}_1 \exp(iK_1 \cdot \bar{r}^*_1)\,\mathrm{d}K_1 \tag{4.17}$$

where $\bar{\mathbf{r}}^*_s = (0, \bar{r}^*_2, \bar{r}^*_3)$ is the component of the position vector $\bar{\mathbf{r}}^*$ of the particle in the plane $\bar{r}_1 = 0$.

The substitution of this expression into [4.16] gives the value of $\bar{\Gamma}$ as

$$\bar{\Gamma}(\mathbf{K}) = \frac{K^2\mathbf{I} - \mathbf{KK}}{K^4} \cdot \mathbf{B} \cdot \mathbf{F}\exp(i\mathbf{K}_s \cdot \bar{r}^*_s), \tag{4.18}$$

where

$$\mathbf{B} = \frac{i}{\pi}\int_{-\infty}^{+\infty} \mathbf{T}^{(3)}(\mathbf{K}) \cdot \mathbf{e}_1 \exp(iK_1\bar{r}^*_1)\,\mathrm{d}K_1. \tag{4.19}$$

This integral can be evaluated and expressed in the form

$$\mathbf{B} = -[\mathbf{I} + K_s\bar{r}^*_1(\mathbf{e}_1 - i\hat{\mathbf{K}}_s)(\mathbf{e}_1 - i\hat{\mathbf{K}}_s)]\exp(-K_s\bar{r}^*_1), \tag{4.20}$$

where

$$K_s = \sqrt{(K_2^2 + K_3^2)},$$

$$\hat{\mathbf{K}}_s = \frac{K_2\mathbf{e}_2 + K_3\mathbf{e}_3}{K_s} \text{ is a unit vector.}$$

Thus one can write

$$\bar{\Gamma}(\mathbf{K}) = -\frac{K^2\mathbf{I} - \mathbf{KK}}{K^4} \cdot [\mathbf{I} + K_s\bar{r}^*_1(\mathbf{e}_1 - i\hat{\mathbf{K}}_s)(\mathbf{e}_1 - \mathrm{i}\mathbf{K}_s)] \cdot \mathbf{F}\exp(-K_s\bar{r}^*_1 + i\mathbf{K}_s \cdot \bar{\mathbf{r}}^*_s). \tag{4.21}$$

The inverse transform gives the velocity of disturbance flow $\bar{\mathbf{u}}(\tilde{\mathbf{r}})$ produced by the wall $\tilde{r}_1 = 0$ as

$$\bar{\mathbf{u}}(\tilde{\mathbf{r}}) = -\frac{1}{8\pi^3}\int \frac{K^2\mathbf{I}-\mathbf{KK}}{K^4}\cdot[\mathbf{I}+K_s\tilde{r}_1^*(\mathbf{e}_1 - i\hat{\mathbf{K}}_s)(\mathbf{e}_1 - i\hat{\mathbf{K}}_s)]\cdot\mathbf{F} \times \exp\{-i\mathbf{K}\cdot(\tilde{\mathbf{r}}-\tilde{\mathbf{r}}_s^*) - K_s\tilde{r}_1^*\}\,\mathrm{d}\mathbf{K}. \tag{4.22}$$

It may be noted that this integral may be evaluated by first integrating with respect to K_1 using contour integration and then with respect to the other variables by transforming to polar coordinates (by writing $K_2 = \rho\cos\phi$, $K_3 = \rho\sin\phi$) in which case one obtains the disturbance flow due to the wall for $\tilde{r}_1 > 0$ as given by Oseen (1927), namely

$$\bar{\mathbf{u}} = -\frac{1}{8\pi}\left\{\frac{\mathbf{I}}{R}+\frac{1}{R^3}(\tilde{\mathbf{r}}-\tilde{\mathbf{r}}^*)(\tilde{\mathbf{r}}-\tilde{\mathbf{r}}^*)+2\tilde{r}_1{}^3\tilde{r}_1\frac{\mathbf{I}}{R^3} + \frac{6\tilde{r}_1^*\tilde{r}_1}{R^5}[(\tilde{r}_1+\tilde{r}_1^*)\mathbf{e}_1+(\tilde{\mathbf{r}}_s-\tilde{\mathbf{r}}_s^*)][(\tilde{r}_1+\tilde{r}_1^*)\mathbf{e}_1-(\tilde{\mathbf{r}}_s-\tilde{\mathbf{r}}_s^*)]\right\}\cdot\mathbf{F}, \tag{4.23}$$

where

$$R = \sqrt{[(\tilde{r}_1+\tilde{r}_1^*)^2+(\tilde{r}_2-\tilde{r}_2^*)^2+(\tilde{r}_3+\tilde{r}_3^*)^2]}$$

is the distance from $\tilde{\mathbf{r}}$ to the image point of $\tilde{\mathbf{r}}^*$ in the plane $\tilde{r}_1 = 0$.

From [3.11], the Fourier transform $\hat{\mathbf{\Gamma}}$ of the velocity field $\hat{\mathbf{u}}$ is given by

$$\hat{\mathbf{\Gamma}} = \bar{\mathbf{\Gamma}} + \mathbf{\Gamma}^*, \tag{4.24}$$

where $\bar{\mathbf{\Gamma}}$ and $\mathbf{\Gamma}^*$ are given by [4.18] and [4.6a] respectively. Thus

$$\hat{\mathbf{\Gamma}} = \frac{K^2\mathbf{I}-\mathbf{KK}}{K^4}\cdot[\mathbf{I}\exp(iK_1\tilde{r}_1^*)+\mathbf{B}]\cdot\mathbf{F}\exp(i\mathbf{K}_s\cdot\tilde{\mathbf{r}}_s^*) \tag{4.25}$$

where $\mathbf{B}$ is given by [4.20]. The flow field $\hat{\mathbf{u}}$ is thus obtained as

$$\hat{\mathbf{u}}(\tilde{\mathbf{r}}) = \frac{1}{8\pi^3}\int\left(\frac{K^2\mathbf{I}-\mathbf{KK}}{K^4}\right)\cdot[\mathbf{I}\exp(iK_1\tilde{r}_1^*)+\mathbf{B}]\cdot\mathbf{F}\exp\{-\mathbf{K}(\tilde{\mathbf{r}}-\tilde{\mathbf{r}}_s^*)\}\,\mathrm{d}\mathbf{K}. \tag{4.26}$$

Since this flow field is identical to $\mathbf{u}$ in the region $\tilde{r}_1 > 0$ it is the flow field produced by the point force $\mathbf{F}$ acting at the position $\tilde{\mathbf{r}}^*$ and satisfying the no slip boundary condition ($\mathbf{u} = 0$) on $\tilde{r}_1 = 0$. Thus from [2.17], it is seen that the Green's function $V_{ij}(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}^*)$ is given by

$$\mathbf{V} = \frac{1}{8\pi^3}\int\left(\frac{K^2\mathbf{I}-\mathbf{KK}}{K^4}\right)\cdot[\mathbf{I}\exp(iK_1\tilde{r}_1^*)+\mathbf{B}]\exp\{-i\mathbf{K}\cdot(\tilde{\mathbf{r}}-\tilde{\mathbf{r}}_s^*)\}\,\mathrm{d}\mathbf{K}, \tag{4.27}$$

the flow field

$$\mathbf{u}(\tilde{\mathbf{r}}) = \mathbf{V}(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}^*)\cdot\mathbf{F}$$

being automatically zero in the region $\tilde{r}_1 < 0$.

If $\Gamma_{ij}(\mathbf{K}, \tilde{\mathbf{r}}^*)$ is the Fourier transform of the Green's function $V_{ij}(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}^*)$ so that

$$\mathbf{\Gamma} = \int \mathbf{V}\exp(i\mathbf{K}\cdot\tilde{\mathbf{r}})\,\mathrm{d}\tilde{\mathbf{r}} \quad\text{and}\quad \mathbf{V} = \frac{1}{8\pi^3}\int\mathbf{\Gamma}\exp(-i\mathbf{K}\cdot\tilde{\mathbf{r}})\,\mathrm{d}\mathbf{K}, \tag{4.28}$$

then

$$\mathbf{\Gamma} = \left(\frac{K^2\mathbf{I} - \mathbf{KK}}{K^4}\right) \cdot [\mathbf{I} \exp(iK_1\tilde{r}_1^*) + \mathbf{B}] \exp(i\mathbf{K}_s \cdot \tilde{\mathbf{r}}_s). \quad [4.29]$$

The integral in [4.27] may be evaluated in a manner similar to that indicated for [4.22] to obtain

$$\mathbf{V}(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}^*) = \frac{1}{8\pi}\left\{\left[\frac{\mathbf{I}}{|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^*|} + \frac{(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^*)(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^*)}{|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^*|^3}\right] - \left[\frac{\mathbf{I}}{R} + \frac{(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^*)(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^*)}{R^3}\right.\right.$$

$$\left.\left. + 2\tilde{r}_1^*\tilde{r}_1\frac{\mathbf{I}}{R^3} + \frac{6\tilde{r}_1^*\tilde{r}_1}{R^5}[(\tilde{r}_1 + \tilde{r}_1^*)\mathbf{e}_1 + (\tilde{\mathbf{r}}_s - \tilde{\mathbf{r}}_s^*)][(\tilde{r}_1 + \tilde{r}_1^*)\mathbf{e}_1 - (\tilde{\mathbf{r}}_s - \tilde{\mathbf{r}}_s^*)]\right]\right\}$$

$$\text{for} \quad \tilde{r}_1 > 0, \quad [4.30a]$$

and

$$\mathbf{V}(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}^*) = \mathbf{0} \quad \text{for} \quad \tilde{r}_1 < 0, \quad [4.30b]$$

so that we have verified that in the region $\tilde{r}_1 > 0$ the Green's function $\mathbf{V}(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}^*)$ is expressed the sum of two terms, the first representing the flow produced in an unbounded medium by a point force at $\tilde{\mathbf{r}} = \tilde{\mathbf{r}}^*$ and the second the disturbance field produced by the plane wall at $\tilde{r}_1 = 0$ (see [4.23]). Furthermore we have also verified that the Green's function $\mathbf{V}(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}^*)$ is identically zero for $\tilde{r}_1 < 0$. It is also observed from [4.30a] that

$$\mathbf{V}(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}^*) = 0(\tilde{r}^{-2}) \quad \text{as} \quad \tilde{r} \to \infty \quad [4.31]$$

where $\tilde{r} = |\tilde{\mathbf{r}}|$. As mentioned in section 2, this result that the creeping motion flow produced by a point force in the neighbourhood of a plane wall dies away like the inverse square of the distance is an important requirement for the direct use of the results of Cox & Brenner (1968).

5. CALCULATION OF LATERAL MIGRATION VELOCITY

In order to evaluate the migration velocity of a sphere given by the [2.19] to [2.21], we must calculate the integrals involved in the expressions for f_1, f_2, g and h given by [2.22]–[2.25]. These integrals may be taken over the whole of space (denoted by $\bar{\Gamma}$) rather than just over the region $\tilde{r}_1 > 0$ (denoted by Γ) if the forms of the Green's function V_{ij} calculated in the previous section are used since V_{ij} is identically zero for $\tilde{r}_1 < 0$. Thus, for example, the quantity h may be written as

$$h = \int_{\bar{\Gamma}} \frac{\partial V_{i2}}{\partial \tilde{r}_2} V_{i1}\, d\tilde{\mathbf{r}}. \quad [5.1]$$

Since $\Gamma_{ij}(\mathbf{K}, \tilde{\mathbf{r}}^*)$ has been written for the Fourier transform of V_{ij}, it follows that the Fourier transforms of the quantities $\partial V_{i2}/\partial \tilde{r}_2$ and V_{i1} appearing in [5.1] are $-iK_2\Gamma_{i2}$ and Γ_{i1} respectively. Now if $F(\mathbf{K})$ and $G(\mathbf{K})$ are the three dimensional Fourier transforms of $f(\tilde{\mathbf{r}})$ and $g(\tilde{\mathbf{r}})$ respectively, the Convolution Theorem states that

$$\int_{\bar{\Gamma}} f(\tilde{\mathbf{r}})g(\tilde{\mathbf{r}})\, d\tilde{\mathbf{r}} = \frac{1}{8\pi^3}\int F(-\mathbf{K})G(\mathbf{K})\, d\mathbf{K},$$

where the integral over $\mathbf{K}$ is taken over the whole of the $\mathbf{K}$-space. Thus applying this result to [5.1], one obtains

$$h = \frac{1}{8\pi^3}\int -iK_2\Gamma_{i2}(\mathbf{K})\Gamma_{i1}(-\mathbf{K})\, d\mathbf{K}. \quad [5.2]$$

Substituting [2.29] and [2.30] for the undisturbed flow field into [2.23]–[2.25] for f_1, f_2 and g and noting that the Fourier transforms of $(\partial V_{ij}/\partial \bar{r}_2)$, $\bar{r}_1 V_{ij}$, $\bar{r}_1(\partial V_{ij}/\partial \bar{r}_2)$ and $\bar{r}_1^2(\partial V_{ij}/\partial \bar{r}_2)$ are respectively $-iK_2\Gamma_{ij}$, $-i(\partial/\partial K_1)\Gamma_{ij}$, $-K_2(\partial/\partial K_1)\Gamma_{ij}$ and $+iK_2(\partial^2/\partial K_1^2)\Gamma_{ij}$, the above analysis for h may be repeated for the integrals f_1, f_2 and g to give

$$g = \beta(2J_1 - \beta J_2), \qquad [5.3]$$

$$f_1 = 2\beta^2(1-\beta)(2K_1 - \beta K_2), \qquad [5.4]$$

and

$$f_2 = 2\beta^2(1-\beta)(2\bar{K}_1 - \beta \bar{K}_2), \qquad [5.5]$$

where

$$J_1 = \frac{1}{8\pi^3}\int\left[iK_2\Gamma_{i1}(-\mathbf{K})\left(1 + i\frac{\partial}{\partial K_1}\right)\Gamma_{i2}(\mathbf{K}) + \Gamma_{21}(-\mathbf{K})\Gamma_{12}(\mathbf{K})\right]d\mathbf{K}, \qquad [5.6]$$

$$J_2 = \frac{1}{8\pi^3}\int\left[iK_2\Gamma_{i1}(-\mathbf{K})\left(1 + \frac{\partial^2}{\partial K_1^2}\right)\Gamma_{i2}(\mathbf{K}) - 2i\Gamma_{21}(-\mathbf{K})\frac{\partial}{\partial K_1}\Gamma_{12}(\mathbf{K})\right]d\mathbf{K}, \qquad [5.7]$$

$$K_1 = \frac{1}{8\pi^3}\int\left[iK_2\Gamma_{i1}(-\mathbf{K})\left(1 + i\frac{\partial}{\partial K_1}\right)\frac{\partial}{\partial \bar{r}_1^*}\Gamma_{i2}(\mathbf{K}) + \Gamma_{21}(-\mathbf{K})\frac{\partial}{\partial \bar{r}_1^*}\Gamma_{12}(\mathbf{K})\right]d\mathbf{K}, \qquad [5.8]$$

$$K_2 = \frac{1}{8\pi^3}\int\left[iK_2\Gamma_{i1}(-\mathbf{K})\left(1 + \frac{\partial^2}{\partial K_1^2}\right)\frac{\partial}{\partial \bar{r}_1^*}\Gamma_{i2}(\mathbf{K}) - 2i\Gamma_{21}(-\mathbf{K})\frac{\partial^2}{\partial K_1\partial \bar{r}_1^*}\Gamma_{12}(\mathbf{K})\right]d\mathbf{K}, \qquad [5.9]$$

$$\bar{K}_1 = \frac{1}{8\pi^3}\int\left[-K_2^2\Gamma_{i1}(-\mathbf{K})\left(1 + i\frac{\partial}{\partial K_1}\right)\Gamma_{i1}(\mathbf{K}) + iK_2\Gamma_{21}(-\mathbf{K})\Gamma_{11}(\mathbf{K})\right]d\mathbf{K}, \qquad [5.10]$$

$$\bar{K}_2 = \frac{1}{8\pi^3}\int\left[-K_2^2\Gamma_{i1}(-\mathbf{K})\left(1 + \frac{\partial^2}{\partial K_1^2}\right)\Gamma_{i1}(\mathbf{K}) + 2K_2\Gamma_{21}(-\mathbf{K})\frac{\partial}{\partial K_1}\Gamma_{11}(\mathbf{K})\right]d\mathbf{K}, \qquad [5.11]$$

all the quantities Γ_{ij} and its derivatives appearing in these integrals being evaluated at the particle (i.e. at $\bar{\mathbf{r}}^* = (1,0,0)$). In deriving [5.10] and [5.11] from [2.25], use was made of the fact that $\bar{r}_2$ and $\bar{r}_2^*$ only appear in [4.30] for V_{ij} in the combination $\bar{r}_2 - \bar{r}_2^*$ so that $(\partial V_{ij}/\partial \bar{r}_2^*) = -(\partial V_{ij}/\partial \bar{r}_2)$ and hence the Fourier transforms of $\partial V_{ij}/\partial \bar{r}_2^*$ and $\partial^2 V_{ij}/\partial \bar{r}_2 \partial \bar{r}_2^*$ are $+iK_2\Gamma_{ij}$ and $+K_2^2\Gamma_{ij}$ respectively.

From the value of $\mathbf{\Gamma}$ given by [4.29] and of $\mathbf{B}$ given by [4.20] it is seen that

$$\Gamma_{i1}(\mathbf{K}) = \frac{K^2\delta_{i1} - K_iK_1}{K^4}[e^{iK_1\bar{r}_1^*} - e^{-K_s\bar{r}_1^*}) - \bar{r}_1^*(K_s + iK_1)e^{-K_s\bar{r}_1^*}]e^{i\mathbf{K}_s\cdot\mathbf{r}_s^*}, \qquad [5.12]$$

$$\Gamma_{i2}(\mathbf{K}) = \left\{\frac{K^2\delta_{i2} - K_iK_2}{K^4}(e^{iK_1\bar{r}_1^*} - e^{-K_s\bar{r}_1^*}) + \frac{K^2\delta_{i1} - K_iK_1}{K^4}\left(i\frac{K_2\bar{r}_1^*}{K_s}\right)\right.$$
$$\left.\times(K_s + iK_1)e^{-K_s\bar{r}_1^*}\right\}e^{i\mathbf{K}_s\cdot\mathbf{r}_s^*}, \qquad [5.13]$$

and

$$\frac{\partial\Gamma_{i2}(\mathbf{K})}{\partial \bar{r}_1^*} = \left\{\frac{K^2\delta_{i2} - K_iK_2}{K^4}(iK_1e^{iK_1\bar{r}_1^*} + K_se^{-K_s\bar{r}_1^*}) + \frac{K^2\delta_{i1} - K_iK_1}{K^4}\left(i\frac{K_2}{K_s}\right)(K_s + iK_1)\right.$$
$$\left.\times(1 - K_s\bar{r}_1^*)e^{-K_s\bar{r}_1^*}\right\}e^{i\mathbf{K}_s\cdot\mathbf{r}_s^*}, \qquad [5.14]$$

which evaluated at $\bar{\mathbf{r}}^* = (1,0,0)$ give

$$\Gamma_{i1}(\mathbf{K}) = \frac{K^2\delta_{i1} - K_iK_1}{K^4}[e^{iK_1} - (1 + K_s + iK_1)e^{-K_s}], \qquad [5.15]$$

$$\Gamma_{i2}(\mathbf{K}) = \frac{K^2\delta_{i2} - K_iK_2}{K^4}(e^{iK_1} - e^{-K_s}) + \frac{K^2\delta_{i1} - K_iK_1}{K^4}\left(\frac{iK_2}{K_s}\right)(K_s + iK_1)\,e^{-K_s}, \qquad [5.16]$$

and

$$\frac{\partial \Gamma_{i2}(\mathbf{K})}{\partial \bar{r}_1^*} = \frac{K^2\delta_{i2} - K_iK_2}{K^4}(iK_1\,e^{iK_1} + K_s\,e^{-K_s}) + \frac{K^2\delta_{i1} - K_iK_1}{K^4}\left(\frac{iK_2}{K_s}\right)(K_s + iK_1)(1 - K_s)\,e^{-K_s}. \qquad [5.17]$$

Substituting these values into the integrals [5.2] and [5.6]–[5.11] and evaluating in a manner similar to that described for the integral [4.22], one obtains

$$h = \frac{1}{64\pi} \qquad [5.18]$$

$$J_1 = \frac{11}{384\pi} \qquad J_2 = \frac{35}{256\pi} \qquad [5.19]$$

$$K_1 = \frac{13}{768\pi} \qquad K_2 = \frac{37}{384\pi} \qquad [5.20]$$

$$\bar{K}_1 = \frac{3}{256\pi} \qquad \bar{K}_2 = \frac{3}{32\pi} \qquad [5.21]$$

Thus by [5.3]–[5.5],

$$h = \frac{1}{64\pi}, \qquad [5.22]$$

$$g = \frac{1}{768\pi}(44 - 105\beta)\beta, \qquad [5.23]$$

$$f_1 = \frac{1}{192\pi}\beta^2(1 - \beta)(13 - 37\beta), \qquad [5.24]$$

and

$$f_2 = \frac{3}{64\pi}\beta^2(1 - \beta)(1 - 4\beta). \qquad [5.25]$$

Therefore the values of the dimensional migration velocity for the particle for the three cases discussed in Section 2 may be expressed as:

(1) For a quiescent or nearly quiescent fluid for which $|V'_\infty/V| \gg 1$

$$u_M^{(1)} = \frac{3}{32}\frac{a(V'_\infty)^2}{\nu}. \qquad [5.26]$$

(2) For a non-neutrally buoyant particle for which $\kappa^2 \ll |V'_\infty/V| \ll 1$

$$u_M^{(2)} = -\frac{1}{128}\frac{aV'_\infty V}{\nu}\beta(44 - 105\beta). \qquad [5.27]$$

(3) For a neutrally buoyant particle for which $|V'_\infty/V| \ll \kappa^2$

$$u_M^{(3)} = +\frac{5}{288}\frac{aV^2}{\nu}\kappa^2\beta^2(1 - \beta)(22 - 73\beta) \qquad [5.28]$$

for a sphere free to rotate, and

$$u_M^{(3)} = \frac{1}{144}\frac{aV^2}{\nu}\kappa^2\beta^2(1-\beta)(61-184\beta) \quad [5.29]$$

for a sphere prevented from rotating.

As shown by Cox & Brenner (1968), the migration velocity for intermediate cases may be obtained by merely adding the corresponding equations for u_M. Thus for example the situation for which $|V'_\infty/V|$ is of order unity [so that we have a situation intermediate between cases (1) and (2)] gives rise to a migration velocity of $u_M^{(1)} + u_M^{(2)}$.
i.e. For $|V'_\infty/V| \sim 1$ and $\kappa \ll 1$

$$u_M = \frac{3}{32}\frac{a(V'_\infty)^2}{\nu} - \frac{1}{128}\frac{aV'_\infty V}{\nu}\beta(44-105\beta). \quad [5.30]$$

Similarly the case intermediate between (2) and (3) for which $\kappa^2 \sim |V'_\infty/V| \ll 1$ gives rise to a migration velocity of

$$u_M = -\frac{1}{128}\frac{aV'_\infty V}{\nu}\beta(44-105\beta) + \frac{5}{288}\frac{aV^2}{\nu}\kappa^2\beta^2(1-\beta)(22-73\beta) \quad [5.31]$$

for a sphere free to rotate, and

$$u_M = -\frac{1}{128}\frac{aV'_\infty V}{\nu}\beta(44-105\beta) + \frac{1}{144}\frac{aV^2}{\nu}\kappa^2\beta^2(1-\beta)(61-184\beta) \quad [5.32]$$

for a sphere prevented from rotating.

It is observed that for case (1) the migration velocity $u_M^{(1)}$ (given by [5.26]) is very much larger than the expressions $u_M^{(2)}$ and $u_M^{(3)}$ (given by [5.27] and [5.28]) while similarly for case (2), $u_M^{(2)}$ is much larger than $u_M^{(3)}$ and $u_M^{(1)}$ and for case (3), $u_M^{(3)}$ is much larger than $u_M^{(1)}$ and $u_M^{(2)}$. Also for the intermediate situation between cases (1) and (2), the migration velocity given by [5.30] is much larger than $u_M^{(3)}$, while similarly the intermediate situation between cases (2) and (3) gives rise to a migration velocity given by [5.30] (or [5.31]) which is much larger than $u_M^{(1)}$. Thus the migration velocity applicable to all cases (including intermediate cases) may be written as

$$u_M = u_M^{(1)} + u_M^{(2)} + u_M^{(3)}, \quad [5.33]$$

so that

$$u_M = \frac{3}{32}\frac{a(V'_\infty)^2}{\nu} - \frac{1}{128}\frac{aV'_\infty V}{\nu}\beta(44-105\beta) + \frac{5}{288}\frac{aV^2}{\nu}\kappa^2\beta^2(1-\beta)(22-73\beta) \quad [5.34]$$

for a sphere free to rotate, and

$$u_M = \frac{3}{32}\frac{a(V'_\infty)^2}{\nu} - \frac{1}{128}\frac{aV'_\infty V}{\nu}\beta(44-105\beta) + \frac{1}{144}\frac{aV^2}{\nu}\kappa^2\beta^2(1-\beta)(61-184\beta) \quad [5.35]$$

for a sphere prevented from rotating.

Furthermore the migration velocity given by [5.30] may thus be taken as being applicable to cases (1) and (2) as well as to the intermediate situation [case (1–2)] between (1) and (2). Also likewise the migration velocities given by [5.31] and [5.32] may be taken as being applicable to cases (2) and (3) as well as to the intermediate situation [cases (2–3)] between (2) and (3).

6. RESULTS AND DISCUSSION

It is seen that the migration velocity given by [5.34] and [5.35] for a sedimenting spherical particle in the flow over the wall may be written in the non-dimensional form

$$u'_M = \frac{3}{32}|\chi|^{-1} - \frac{1}{128}\beta(44-105\beta)\,\mathrm{sgn}\,\chi + \frac{5}{288}|\chi|\kappa^2\beta^2(1-\beta)(22-73\beta) \qquad [6.1]$$

for a sphere free to rotate, and

$$u'_M = \frac{3}{32}|\chi|^{-1} - \frac{1}{128}\beta(44-105\beta)\,\mathrm{sgn}\,\chi + \frac{1}{144}|\chi|\kappa^2\beta^2(1-\beta)(61-184\beta) \qquad [6.2]$$

for a sphere prevented from rotating where u'_M is the dimensionless migration velocity defined as

$$u'_M = u_M \Big/ \left|\frac{aVV'_\infty}{\nu}\right|, \qquad [6.3]$$

and χ is the ratio V/V'_∞ of the flow velocity V at $r'_1 = l$ to the sedimentation velocity V'_∞ taken as being positive in the r'_2-direction. Thus the dimensionless velocity only depends on the quantities χ, κ (the ratio a/d of particle radius a to the particle to wall distance d) and β equals to d/l determining the particle position in relation to the flow.

For the conditions under which [5.30] is valid, namely for cases (1), (1–2) and (2) [i.e. for $|\chi|^{-1} \gg \kappa^2$], the dimensionless velocity u'_M may be written as

$$u'_M = \frac{3}{32}|\chi|^{-1} - \frac{1}{128}\beta(44-105\beta)\,\mathrm{sgn}\,\chi \qquad [6.4]$$

and is thus a function only of χ and β. This result has been plotted in figure 3 in which lines of constant u'_M have been drawn on a (χ, β) diagram. Regions for which u'_M is negative represent particle migration towards the wall while u'_M positive represents migration away from the wall. Points on the line $u'_M = 0$ represents equilibrium positions of the particle for which there is no migration velocity. These are stable for $(\partial u'_M/\partial\beta) < 0$ and unstable for $(\partial u'_M/\partial\beta) > 0$. It is observed from figure 3 that:

(i) For case (1), $\chi = 0$, the migration of the particle is always away from the wall and has a magnitude which is independent of its position (see [5.26]) although it must be remembered that the value of d, the distance of the sphere from the wall, cannot be made indefinitely large without invalidating the condition [2.7].

(ii) For case (2) with a particle less dense than the fluid in an upflow (or more dense than the fluid in a downflow) so that $\chi = +\infty$, it is seen that ([5.27]) particle migration is towards the wall for the particle near the wall with $\beta < (44/105)$ $(= 0.419)$. This agrees with the experimental results of Repetti & Leonard (1964) for two dimensional Poiseuille flow. However the result that for $\beta > 0.419$ migration should be away from the wall does not agree with the above experiments presumably because at these larger values of β, the influence of the different wall configuration in the experiments is being felt.

(iii) For case (2) with a particle less dense than the fluid in a downflow (or more dense than the fluid in an upflow) so that $\chi = -\infty$, it is seen that ([5.27]) the migration is just the opposite to that described above in (ii) so that the direction of migration is away from the wall (and is in agreement with experiment) for $\beta < 0.419$ but is in disagreement with experiment for $\beta > 0.419$, presumably due to the effects of other walls present.

(iv) For a particle less dense than the fluid in an upflow (or more dense than the fluid in a downflow), the migration is always away from the wall for $\chi < 2.6$ (see figure 3), but there

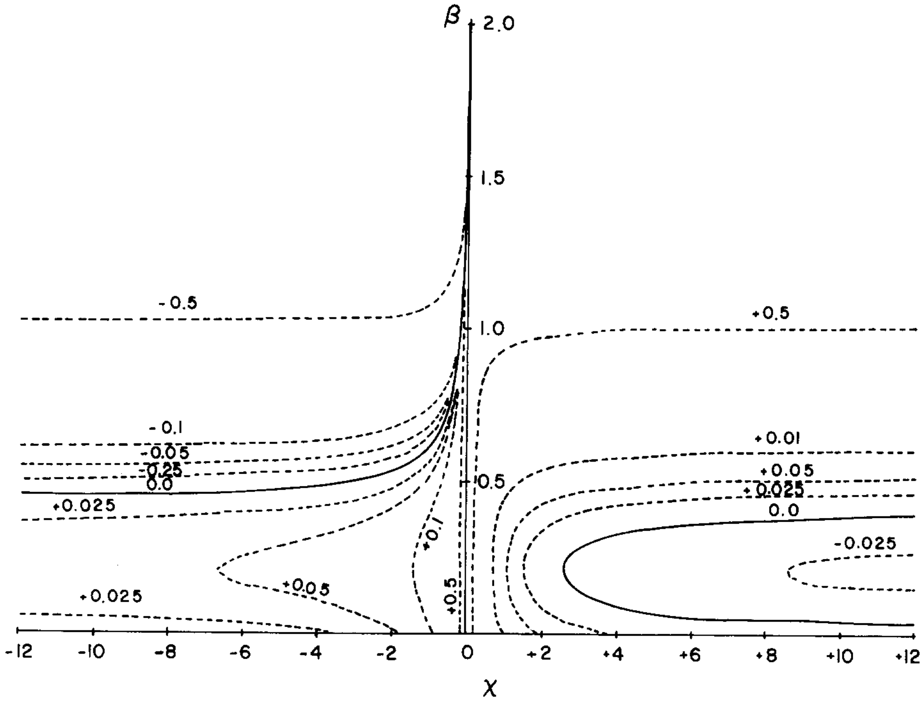

Figure 3. Lines of constant u'_M drawn on (χ, β) diagram for the situation $|\chi|^{-1} \gg \kappa^2$ [i.e. for cases (1), (1–2) and (2)]. The line representing equilibrium positions of particle ($u'_M = 0$) is indicated thus ——.

exist two equilibrium positions for $\chi > 2.6$. Of these, the equilibrium position closer to the wall (with $\beta = 0.21$ at $\chi = 2.6$ decreasing with $\beta \to 0$ as $\chi \to \infty$) is stable while the outer equilibrium position (with $\beta = 0.21$ at $\chi = 2.6$ increasing with $\beta \to 0.419$ as $\chi \to \infty$) is unstable. While it is unlikely that at large values of χ that these equilibrium positions would be observable (since the inner equilibrium position would then be so close to the wall that its distance from the wall could not be made much larger than a and the outer equilibrium position would not exist due to effect of other walls), it might be possible to observe such equilibrium positions at values of χ slightly larger than 2.6.

(v) For a particle less dense than the fluid in a downflow (or more dense than the fluid in an upflow), the migration is towards a stable equilibrium position which is at $\beta = \infty$ for $\chi = 0$ and decreases with $\beta \to 0.419$ as $\chi \to -\infty$. As noted in (iii) above, the inward migration towards the wall at the larger values of β is unlikely to be observed in practice.

(vi) The equilibrium positions (whether stable or unstable) depend only on the value of $\chi = V/V'_\infty$ and are therefore not dependent directly on the particle radius a, although of course in general a change in particle size will cause a change in the value of χ.

For the conditions under which [5.31] and [5.32] are valid, namely for cases (2), (2–3) and (3) [i.e. for $|\chi|^{-1} \ll 1$], the dimensionless velocity u'_M may be written as

$$u'_M = -\frac{1}{128}\beta(44 - 105\beta)\,\mathrm{sgn}\,\psi + \frac{5}{288}|\psi|^{-1}(1-\beta)(22-73\beta) \qquad [6.5]$$

for a sphere free to rotate, and

$$u'_M = -\frac{1}{128}\beta(44 - 105\,\beta)\,\mathrm{sgn}\,\psi + \frac{1}{144}|\psi|^{-1}(1-\beta)(61-184\,\beta) \qquad [6.6]$$

for a sphere prevented from rotating, where ψ is defined as

$$\psi = \chi^{-1}\kappa^{-2}\beta^{-2} = \frac{V'_\infty}{V}\frac{l^2}{a^2}. \qquad [6.7]$$

Thus u'_M is a function only of this quantity ψ and of β. These results [6.5] and [6.6] have been plotted respectively in figures 4a and 4b in which lines of constant u'_M have been drawn on a (ψ, β) diagram. Again the equilibrium positions of the particle are indicated by the line $u'_M = 0$, while the stability of such positions are determined by the sign of $\partial u'_M/\partial\beta$. It is observed from figures 4a and 4b that:

(i) For a neutrally buoyant particle ($\psi = 0$) the migration is away from the wall at small distances from the wall ($0 < \beta < (22/73)$ for particles free to rotate and $0 < \beta < (61/184)$ for particles prevented from rotating) and also at positions beyond the point of maximum flow velocity ($1 < \beta$). In the intermediate positions ($(22/73) < \beta < 1$ for particles free to rotate and $(61/184) < \beta < 1$ for particles prevented from rotating) the migration is towards the wall. Thus there is a position of stable equilibrium at $\beta = (22/73) = 0.3014$ for particles free to rotate and at $\beta = (61/184) = 0.3315$ for particles prevented from rotating while in either case there is a position of unstable equilibrium at the position $\beta = 1$ of maximum flow velocity. These results agree qualitatively with experiments performed with neutrally buoyant particles in flow between plane parallel walls (Repetti & Leonard 1964; Yanizeski 1968; Tachibana 1973) and in tube flow (Segre & Silberberg 1961, 1962a, b; Oliver 1962, Karnis *et al.* 1966a, b; Jeffrey & Pearson 1965) in that the experiments show a position of stable equilibrium between the wall and the position of maximum velocity (at about $\beta = 0.4$–0.5) and a position of unstable equilibrium at the position of maximum flow velocity ($\beta = 1$). Furthermore it was observed by Oliver (1962) that in tube flows a particle which is prevented from rotating migrates to a position farther from the wall than that for a particle free to rotate, a result also in qualitative agreement with the present theory.

(ii) For a particle less dense than the fluid in an upflow (or more dense than the fluid in a downflow) the stable equilibrium position which for a neutrally buoyant particle free to rotate is at $\beta = 0.3014$ [or $\beta = 0.3315$ for a particle prevented from rotating] moves towards

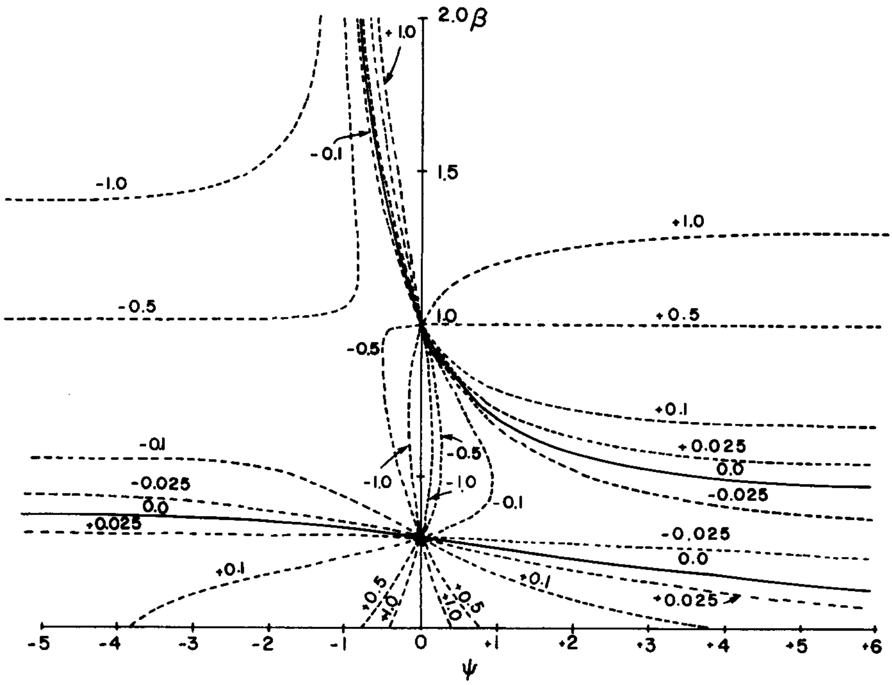

Figure 4a. Lines of constant u'_M drawn on (ψ, β) diagram for a sphere free to rotate under the situation $|\chi|^{-1} \ll 1$ [i.e. for cases (2), (2–3) and (3)]. The line representing equilibrium positions of particle ($u'_M = 0$) is indicated thus ——.

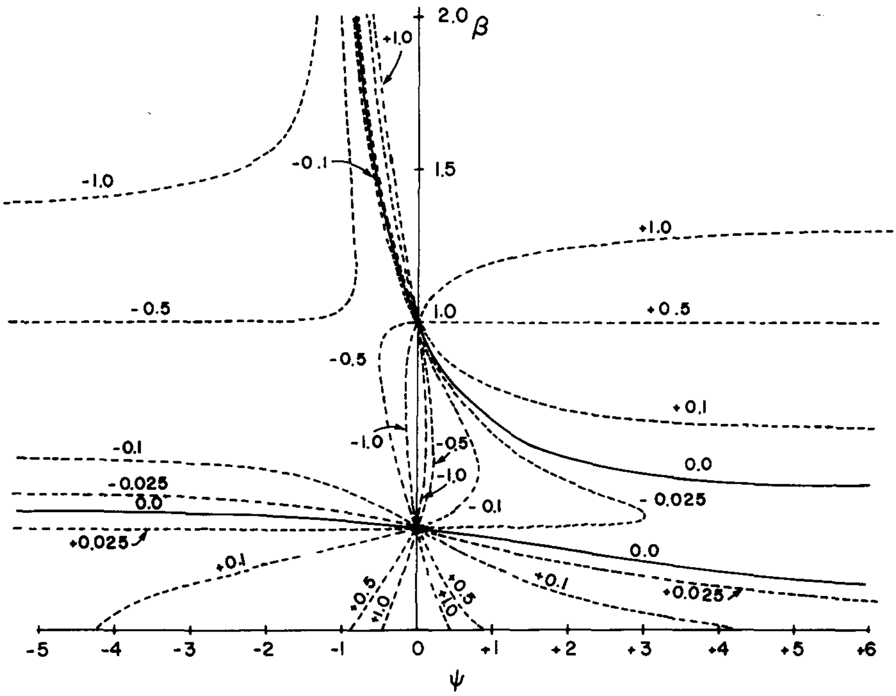

Figure 4b. Same as figure 4a except that sphere is prevented from rotating.

the wall as ψ increases with its position $\beta \to 0$ as $\psi \to \infty$. This result that the stable equilibrium position depends on the density difference between solid and fluid for near neutrally buoyant particles has been observed experimentally by Repetti & Leonard (1964) and Tachibana (1973). The unstable equilibrium position at $\beta = 1$ for a neutrally buoyant particle is observed from figures 4a, b to move towards the wall as ψ increases with $\beta \to 0.419$ as $\psi \to \infty$ whether or not the particle is allowed to rotate.

(iii) For a particle less dense than the fluid in a downflow (or more dense than the fluid in an upflow) the position of stable equilibrium at $\beta = 0.3014$ [or $\beta = 0.3315$ for a particle prevented from rotating] moves away from the wall as ψ becomes more negative with $\beta \to 0.419$ as $\psi \to -\infty$ (see Repetti & Leonard 1964). The unstable equilibrium position at $\beta = 1$ for a neutrally buoyant particle also moves away from the wall as ψ becomes more negative with $\beta \to +\infty$ as $\psi \to -\infty$. However this latter result can hardly be expected to be valid for tube flow or flow between parallel plates due to the expected large effect of the wall geometry.

(iv) In the limit of $\psi \to \pm\infty$, the results given in figures 4a and b become identical with those for $\chi \to \pm\infty$ shown in figure 3, each then representing case (2).

(v) The equilibrium positions whether stable or unstable depend only on the value of $\psi = (V'_\infty/V)(l^2/a^2)$ and since for a uniform particle

$$V'_\infty = \frac{2(\Delta\rho)ga^2}{9\mu},$$

where $\Delta\rho$ is density of the particle minus that of the fluid, it follows that

$$\psi = \frac{2(\Delta\rho)gl^2}{9\mu V},$$

so that the equilibrium positions of the particle are seen to depend on $\Delta\rho$ but to be independent of the particle size a.

The results and theory which have been described in this paper provide a means by which the lateral inertial migration of a spherical particle in a flow near a planar (or nearly planar) solid wall may be calculated whether or not there exists a pressure gradient in the undisturbed flow. In general the results can be expected to be valid when β is small, but for the particle far from the wall (β large), the results would only be valid for an experimental situation where the flow is that given by [2.28] over a region $|\mathbf{r}'| < \lambda d$ where $\lambda \gg 1$.

The lateral migration of a particle in rectilinear flow between a pair of vertical parallel walls has been investigated theoretically by Ho & Leal (1974) for the case of a neutrally buoyant particle [case (3)] and by Vasseur (1973) for all the other situations discussed in this paper. These results are shown by Vasseur & Cox (1976) to be in very good agreement with the present results when β is small.

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